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Frame Independent Nonlocality for Three Qubit State

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Abstract Bell's inequality is investigated for the three qubit GHZ state in relativistic regime. Two different relativistic spin operator are considered. One of them is defined by Lee and Ee (New J. Phys. 6:67, 2004). and the other which is the Pauli-Lubanski pseudovector used by Kim and Son (Phys. Rev. A 71:014102, 2005). It is shown that for both spin operator Bell's inequality is still maximally violated in a Lorentz-boosted frame.

Keywords Bell's inequality · Relativistic spin operator

An important resource for quantum communication and computation is entanglement. Recently, there has been much interest in the study of entanglement for inertial observers [1–9]. Ahn et al. [4] calculated the Bell observable for the Bell states under Lorentz boost and showed that the Bell's inequality is not violated in the relativistic limit. They used the relativistic spin observable which is closely related to the spatial components of the Pauli-Lubanski pseudovector [5] and transformed the state under Lorentz boost accordingly. Their results strongly suggested that the entanglement is not preserved under the Lorentz boost. Lee et al. [1] showed that maximal violation of the Bell's inequality can be achieved by properly adjusting the directions of the spin measurement even in a relativistically moving inertial frame. Kim et al. [2] obtained an observer-independent Bell's inequality, so that it is maximally violated as long as it is violated maximally in the rest frame. They showed that the Bell observable and Bell states for Bell's inequality should be transformed following the principle of relativistic covariance, which results in a frame independent Bell's inequality. Moradi et al. [7] studied the Bell's inequality for three-qubit GHZ state in relativistic frame using the Czachor's spin operator. In this paper we would like to investigate the Bell's inequality for three particle states using two spin operators introduced in references [1] and [2] and show that relativistic invariant Bell's inequality can be achieved.

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First, we assume the spin measuring device is fixed in the lab frame and particles are moving with the same velocity in the lab frame The relativistic spin observable used in Ref. [1] is

$$\hat{a} = \frac{\vec{a}_p.\vec{\sigma}}{|\lambda(\vec{a}_p.\vec{\sigma})|}. \tag{1}$$

The measuring axis $\vec{a}_p$ in the moving frame is the spatial part of Lorentz transformed ${a_p}^{\mu} = L_z(-\xi)^{\mu}{}_{\nu}a^{\nu}$ where $\tanh\xi \equiv \beta_p$ is the boost of particles. If three particles move in $+z$ direction we have

$$a_p = (-a_z \sinh\xi, a_x, a_y, a_z \cosh\xi). \tag{2}$$

Then the relativistic spin observable for particle 1 in the present case is

$$\hat{a} = \frac{a_x\sigma_x + a_y\sigma_y + a_z\sigma_z \cosh\xi}{\sqrt{1 + a_z^2 \sinh^2\xi}}, \tag{3}$$

there are the same relations for particles 2 and 3. The expectation value for joint spin measurement of particles can be expressed as

$$\langle \hat{a} \otimes \hat{b} \otimes \hat{c} \rangle = \langle \psi | \frac{\vec{a}_p.\vec{\sigma}}{|\lambda(\vec{a}_p.\vec{\sigma})|} \otimes \frac{\vec{b}_p.\vec{\sigma}}{|\lambda(\vec{b}_p.\vec{\sigma})|} \otimes \frac{\vec{c}_p.\vec{\sigma}}{|\lambda(\vec{c}_p.\vec{\sigma})|} |\psi\rangle, \tag{4}$$

where the state here is GHZ state and given by

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)|\vec{p}_1, \vec{p}_2, \vec{p}_3\rangle, \tag{5}$$

here 0 and 1 represent spins polarized up and down along the z axis. Now, the expectation value of the joint spin measurement (4) over state (5) is given by

$$E(\vec{a}, \vec{b}, \vec{c}) = \frac{a_x b_x c_x - a_x b_y c_y - a_y b_x c_y - a_y b_y c_x}{\sqrt{(1 + a_z^2 \sinh^2\xi)(1 + b_z^2 \sinh^2\xi)(1 + c_z^2 \sinh^2\xi)}}. \tag{6}$$

The Bell's inequality to the case of three particles in terms of correlation functions is as follows

$$\varepsilon = |E(\vec{a}, \vec{b}, \vec{c'}) + E(\vec{a}, \vec{b'}, \vec{c}) + E(\vec{a'}, \vec{b}, \vec{c}) - E(\vec{a'}, \vec{b'}, \vec{c'})| \leq 2. \tag{7}$$

The vector set inducing the maximal violation of Bell's inequality for the GHZ state in the non-relativistic case are

$$\begin{aligned} \vec{a} = \vec{b} = \vec{c} = \hat{y}, \\ \vec{a'} = \vec{b'} = \vec{c'} = \hat{x}. \end{aligned} \tag{8}$$

Using the vector set (8) in (6) we get

$$E(\vec{a}, \vec{b}, \vec{c'}) = E(\vec{a}, \vec{b'}, \vec{c}) = E(\vec{a'}, \vec{b}, \vec{c}) = -E(\vec{a'}, \vec{b'}, \vec{c'}) = -1. \tag{9}$$

Then we have $\varepsilon = 4$, which means that Bell's inequality is dependent of particles velocities and maximally violated.

Now assume observers for particles are sitting in the moving frame Lorentz-boosted with respect to the lab frame in a direction perpendicular to the particles' velocities. Again, we choose that particles are moving in the $+z$ direction in the lab frame, and the moving frame is Lorentz-boosted to the $-x$ direction. In this case the relativistic correlation function is

$$E'(\vec{a},\vec{b},\vec{c}) = \langle \mathrm{GHZ}'|\frac{\vec{A}.\vec{\sigma}\otimes\vec{B}.\vec{\sigma}\otimes\vec{C}.\vec{\sigma}}{|\vec{a}_{\Lambda p}||\vec{b}_{\Lambda p}||\vec{c}_{\Lambda p}|}|\mathrm{GHZ}'\rangle, \tag{10}$$

where

$$\vec{A} = \begin{pmatrix} a_x(\cos^2\theta - \cosh\eta\sin^2\theta) - a_z(1+\cosh\eta)\sin\theta\cos\theta \\ a_y \\ a_x(1+\cosh\eta)\sin\theta\cos\theta - a_z(\sin^2\theta - \cosh\eta\cos^2\theta) \end{pmatrix}, \tag{11}$$

and

$$|\vec{a}_{\Lambda p}| = \sqrt{1+\sinh^2\eta(a_x\sin\theta + a_z\cos\theta)^2}. \tag{12}$$

Here

$$\tan\theta \equiv (E_p\sinh\chi)/p = \frac{\sinh\chi}{\tanh\xi}, \tag{13}$$

where χ and ξ are the boost speed and

$$\tanh\eta \equiv \frac{|\vec{p}_\Lambda|}{E_{\Lambda p}} = \frac{\sqrt{(\tanh^2\xi + \sinh^2\chi)}}{\cosh\chi}. \tag{14}$$

Finally $|\mathrm{GHZ}'\rangle$ is the Lorentz transformed GHZ-state [7]

$$\begin{aligned} |\mathrm{GHZ}'\rangle = {} & \frac{1}{\sqrt{2}}(g_1|000\rangle + g_2|001\rangle + g_3|010\rangle + g_4|011\rangle \\ & + g_5|100\rangle + g_6|101\rangle + g_7|110\rangle + g_8|111\rangle)|\vec{p}_1\vec{p}_2\vec{p}_3\rangle_\Lambda, \end{aligned} \tag{15}$$

where

$$\begin{aligned} g_1 &= \cos^3\frac{\delta}{2} - \sin^3\frac{\delta}{2}, \\ g_2 = g_3 = g_5 &= \sin^2\frac{\delta}{2}\cos\frac{\delta}{2} + \sin\frac{\delta}{2}\cos^2\frac{\delta}{2}, \\ g_4 = g_6 = g_7 &= \sin^2\frac{\delta}{2}\cos\frac{\delta}{2} - \sin\frac{\delta}{2}\cos^2\frac{\delta}{2}, \\ g_8 &= \cos^3\frac{\delta}{2} + \sin^3\frac{\delta}{2}. \end{aligned} \tag{16}$$

Here δ is the Wigner's rotation and defined as

$$\tan\delta \equiv \frac{\sinh\xi\sinh\chi}{\cosh\xi + \cosh\chi}. \tag{17}$$

Now the relativistic correlation function reduces to

$$\begin{aligned}E'(\vec{a},\vec{b},\vec{c}) = {} & \{(1+\sinh^2\eta(a_x\sin\theta+a_z\cos\theta)^2)(1+\sinh^2\eta(b_x\sin\theta+b_z\cos\theta)^2)\\ & \times(1+\sinh^2\eta(c_x\sin\theta+c_z\cos\theta)^2)\}^{-1/2}\\ & \times[A_xB_xC_x(\cos^3\delta)-A_zB_zC_z(\sin^3\delta)\\ & -\sin\delta\cos^2\delta(A_xB_xC_z+A_xB_zC_x+A_zB_xC_x)\\ & +\sin^2\delta\cos\delta(A_xB_zC_z+A_zB_xC_z+A_zB_zC_x)\\ & -\cos\delta(A_xB_yC_y+A_yB_xC_y+A_yB_yC_x)\\ & +\sin\delta(A_zB_yC_y+A_yB_zC_y+A_yB_yC_z)].\end{aligned}$$

Using vector set (8) we get

$$\begin{aligned}E'(\vec{a},\vec{b},\vec{c}') &= E'(\vec{a},\vec{b}',\vec{c}) = E'(\vec{a}',\vec{b},\vec{c})\\ &= \frac{1}{\sqrt{1+\sinh^2\eta\sin^2\theta}}[-\cos\delta(\cos^2\theta-\cosh\eta\sin^2\theta)\\ &\quad+\sin\delta(1+\cosh\eta)\sin\theta\cos\theta],\end{aligned} \tag{18}$$

and

$$\begin{aligned}E'(\vec{a}',\vec{b}',\vec{c}') = {} & \frac{1}{\sqrt{(1+\sinh^2\eta\sin^2\theta)^3}}\\ & \times[\cos\delta(\cos^2\theta-\cosh\eta\sin^2\theta)-\sin\delta(1+\cosh\eta)\sin\theta\cos\theta]^3,\end{aligned} \tag{19}$$

putting (19) and (18) in (9) we arrive at

$$\begin{aligned}\varepsilon' = \Bigg| & \frac{3}{\sqrt{1+\sinh^2\eta\sin^2\theta}}[-\cos\delta(\cos^2\theta-\cosh\eta\sin^2\theta)+\sin\delta(1+\cosh\eta)\sin\theta\cos\theta]\\ & -\frac{1}{\sqrt{(1+\sinh^2\eta\sin^2\theta)^3}}[\cos\delta(\cos^2\theta-\cosh\eta\sin^2\theta)\\ & -\sin\delta(1+\cosh\eta)\sin\theta\cos\theta]^3\Bigg|.\end{aligned} \tag{20}$$

In both cases as $\chi \to 0$ or $\xi \to 0$ the inequality maximally violated. One can show that we have the maximum value for violation of inequality for any reference frame if the following relations are satisfied

$$\frac{\vec{A}_c}{|\vec{a}_{c\Lambda_p}|} = \vec{\tilde{a}}, \qquad \frac{\vec{B}_c}{|\vec{b}_{c\Lambda_p}|} = \vec{\tilde{b}}, \qquad \frac{\vec{C}_c}{|\vec{c}_{c\Lambda_p}|} = \vec{\tilde{c}}, \tag{21}$$

where

$$\vec{\tilde{a}} = R_y(\delta)\vec{a}, \qquad \vec{\tilde{b}} = R_y(\delta)\vec{b}, \qquad \vec{\tilde{c}} = R_y(\delta)\vec{c}, \tag{22}$$

with

$$R_y(\delta) = \begin{pmatrix}\cos\delta & 0 & \sin\delta\\ 0 & 0 & 0\\ -\sin\delta & 0 & \cos\delta\end{pmatrix}. \tag{23}$$

So Bell's inequality is still maximally violated in a Lorentz-boosted frame, if we properly choose new set of spin measurement directions. Here we do the same analysis using the relativistic spin operator introduced in Ref. [2]

$$\hat{O}(a) = \frac{2a^{\mu} W_{\mu}}{mc\hbar}. \tag{24}$$

where the Pauli-Lubanski pseudovector W_{μ} in the rest frame is $(0, m\vec{S})$ and $a^{\mu} = (0, \vec{a})$. The spin vector and the axis should be transformed by the appropriate transformation law. The observable $\hat{O}(a,b,c) = \hat{O}(a) \otimes \hat{O}(b) \otimes \hat{O}(c)$ transforms as

$$\begin{aligned} \hat{O}'(a,b,c) &= U(\Lambda)\hat{O}(a,b,c)U^{-1}(\Lambda) \\ &= (2/mc\hbar)^3 a^{\mu} b^{\nu} c^{\rho} U(\Lambda) W_{\mu} \otimes W_{\nu} \otimes W_{\rho} U^{-1}(\Lambda) \\ &= 8\vec{a}.\vec{S}_R \otimes \vec{b}.\vec{S}_R \otimes \vec{c}.\vec{S}_R, \end{aligned} \tag{25}$$

where $\vec{S}_R = D(W)\vec{S}D^{-1}(W)$, here $D(W)$ is the Wigner representation of the Lorentz group for spin-$\frac{1}{2}$

$$D(W) = \begin{pmatrix} \cos\frac{\delta}{2} & -\sin\frac{\delta}{2} \\ \sin\frac{\delta}{2} & \cos\frac{\delta}{2} \end{pmatrix}. \tag{26}$$

Now the transformation of the spin is rewritten by the transformation of the axes along with the following relation [2]

$$\begin{aligned} 2\vec{a}.\vec{S}_R &= \vec{a}.D(W)\vec{\sigma}D^{-1}(W) = 2\vec{a}_R.\vec{S} \\ &= \begin{pmatrix} a_z\cos\delta - a_x\sin\delta & a_z\sin\delta + a_x\cos\delta - ia_y \\ a_z\sin\delta + a_x\cos\delta + ia_y & -a_z\cos\delta + a_x\sin\delta \end{pmatrix}. \end{aligned} \tag{27}$$

Then we have

$$\hat{O}'(a,b,c) = 8\vec{a}_R.\vec{S} \otimes \vec{b}_R.\vec{S} \otimes \vec{c}_R.\vec{S} = \hat{O}(\vec{a}_R, \vec{b}_R, \vec{c}_R). \tag{28}$$

Under the Wigner rotation, the unit vector $\vec{a}$ is transformed as

$$\vec{a}_R = (a_x\cos\delta + a_z\sin\delta, a_y, -a_x\sin\delta + a_z\cos\delta), \tag{29}$$

after some mathematical manipulations, we get

$$E'(\vec{a}, \vec{b}, \vec{c}) = a_x b_x c_x - (a_x b_y c_y + a_y b_x c_y + a_y b_y c_x). \tag{30}$$

Then the expectation value of GHZ state is invariant under the Lorentz boost. So the maximal violation of Bell's inequality is maintained at any reference frame.

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